OUTLINE OF RESEARCH WORK ENTITLED

Analytical Study of Integro-Differential Equations Involving Fractional Operators

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By

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1 Introduction

Fractional or non-integer calculus deals with derivatives and integrals of arbitrary real or complex order [38]. This subject arisen from a well-known scientific discussion between L'Hopital and Leibniz in 1995, and then, investigated and extended by many renown mathematicians like Euler, Laplace, Abel, Liouville and Riemann [38]. The subject has received attention of many scientists, not only in mathematics, but also in physics and engineering. So it has become a hot issue in recent years. However, fractional calculus extends the notion of derivative for those cases that the derivative order is not integer.

Despite of the idea of fractional derivatives and integrals can be considered as a generalization of corresponding standard ones, it is still quite a strange topic, very hard to explain. Because, unlike commonly used differential operators, it is not related to some important geometrical meaning, such as the trend of functions or their convexity. So, sometimes this mathematical tool could be judged far from reality. But indeed many physical phenomena have intrinsic fractional order description and so fractional order calculus is necessary in order to explain them. It worth noting that there exist only two main definitions of the fractional derivative; the first was proposed by Riemann and Liouville and is the derivative of the convolution of a given function and a power law kernel, the second one was suggested by Caputo and it is the convolution of the local derivative of a given function with power law function. Ongoing debate amount researchers within this field

to distinguish what version is mathematically well defined, this controversy lead many researchers to undertake theoretical and applied studies to clearly establish that definition that is mathematically formulated. Some applied mathematicians suggested that the Caputo is suitable for real-world problem due to the fact that it allows usual initial conditions when playing with integral transform for instance Laplace transform [32, 39, 40, 41, 45].

On the other hand, the Riemann-Liouville definition satisfied all the mathematical principle under the scope of fractional calculus, more importantly when using Laplace transform we obtain initial condition with fractional exponent which is actually realistic in practical and mathematical point of view because we are in the scope of fractional calculus, because of this, must researchers agreed that the Riemann-Liouville definition is more suitable. Although these derivatives are powerful mathematical tools to model real world problems, it is important to note that they cannot account for those real world problems involving continuous probability distribution. Therefore, due to the fact that the power law cannot be used to model all the physical problems, athours have suggested many alternative concept of differentiation using the exponential decay as kernel instead of the power law. However, due to the memory property of fractional derivatives, the theory and applications of fractional calculus has become extremely useful and important in modeling of biological processes, applied mathematics, physics, and engineering [4,5,7]. The orientation of research in modeling real world problems is shifting toward the use of fractional order derivative. Due to this fact, in recent decades,

the use of fractional order derivatives has become extensively attractive in several fields of science and engineering to describe different kinds of problems. The reason for this is that many real-world physical systems display fractional order dynamics and their behavior is governed by fractional systems.

2 Review of Literature and Development in the Subject

Fractional calculus is the most well known and valuable branch of mathematics which gives a good framework for biological and physical phenomena, mathematical modeling of engineering, etc. To get a couple of developments about the theory of fractional differential equations, one can allude to the monographs of Hilfer [27], Kilbas et al [32], Miller and Ross [35], Oldham [37], Pudlubny [38], Sabatier et al [39], Tarasov [41] and the references therein.

A history of the development of fractional differential operators can be found in [1]. The memory effect of these derivatives and their nonlocal property, is one of the main reasons to use them in various applications. During the past decades, FDEs have been playing more and more important roles in many fields such as physics, biology, mechanics and chemistry, etc. [48]. More recently, the applications of FDEs have been extended to quantum mechanics such that fractional quantum mechanics came into being [51,52].

As everyone knows, obtaining analytical solutions for such problems is usually difficult. Therefore, approximate methods for finding the approximate solutions

of these equations are very necessary and useful. In fact, finite difference methods, finite element methods, spectral methods, etc., have been presented to solve the various fractional equations. The review article [3] contains an upto-date bibliography on finite difference methods for solving FDEs until 2012. At present, this field of study is still developing for its many applications in such varied fields as viscoelasticity, fluid flow and hydrology. The development of efficient numerical techniques for approximating the solutions of FDEs has been an important issue in the last decades.

However, it has been observed that most of the work on the topic involves either Riemann-Liouville or Caputo-type fractional derivative. Besides these derivatives, Hadamard fractional derivative is another kind of fractional derivatives that was introduced by Hadamard in 1892 [26]. This fractional derivative differs from the other ones in the sense that the kernel of the integral (in the definition of Hadamard derivative) contains the logarithmic function of arbitrary exponent. Good overviews and applications of where the Hadamard derivative and the Hadamard integral arise can be found in the papers by Butzer et al [19]. Other important results dealing with Hadamard fractional calculus and Hadamard differential equations can be found in [10,15,17,28,35,38,43], as well as in the monograph by Kilbas et al [32]. In recent years, Jarad et al [28] modified the Hadamard fractional derivative into a more suitable one having physical interpretable initial conditions similar to the singles in the Caputo setting and called it Caputo-Hadamard fractional derivative. Details and properties of the modified

derivative can be found in [28]. To the best of our knowledge, few results can be found in the literature concerning boundary value problems for Caputo-Hadamard fractional differential equations [1, 9, 11, 18]. On the other hand, it is well known that the nonlocal condition is more appropriate than the local condition (initial and /or boundary) to describe correctly certain features of applied mathematics and physics such as blood flow problems, chemical engineering, thermo-elasticity, population dynamics and so on [16, 20, 25, 31, 33, 40, 43, 44].

At the present day, there are many results on the existence of solutions for fractional differential equations. For more details, the readers are referred to the previous studies [14, 16, 25, 29, 37] and the references therein.

The importance of fractional derivatives for modeling phenomena in different branches of science and engineering is due to their nonlocality nature, an intrinsic property of many complex systems. Integro-differential equations appear in the mathematical modeling of several real world problems such as, heat transfer phenomena [19], forced-convective flow over a heat-conducting plate [27], etc.

Unlike the derivative of integer order, fractional derivatives do not take into account only local characteristics of the dynamics but considers the global evolution of the system; for that reason, when dealing with certain phenomena, they provide more accurate models of real-world behavior than standard derivatives.

Fractional differential operators describe mechanical and physical processes with historical memory and spatial global correlation and for the basic theory-see [1-3]. Results on existence, stability and controllability for differential equa-

tions with Caputo, Riemann-Liouville and Hilfer type fractional derivatives can be found, for example, in [4-19]. Caputo and Fabrizio introduced a new nonlocal derivative without a singular kernel and Atangana and Nieto and Lopez [36] studied the numerical approximation of this new fractional derivative and established a modified resistance loop capacitance (RLC) circuit model. The authors in [22] presented a fractional integral corresponding to the Caputo-Fabrizio fractional derivative and introduced Caputo-Fabrizio fractional differential equations and established existence and uniqueness results. Jarad et al. [28] extended the study to Caputo-Fabrizio fractional integro-differential equations and obtained the approximate solution. The authors in [24] established a new Kortewegde Vries-Burgers equation involving the Caputo-Fabrizio fractional derivative with no singular kernel and presented existence and uniqueness results and also gave numerical approximations.

Hyers-Ulam stability is a concept that provides an approximate solution for the exact solution in a simple form for integro-differential equations. A Laplace transform method is applied to show the Hyers-Ulam stability for integer order integro-differential equations in [4, 5, 42] and [27] adopted the idea and applied a Laplace transform method to show the Hyers-Ulam stability for fractional order integro-differential equations involving Caputo derivatives. There are many works on the integro-differential equations involving fractional derivatives [28-36]. Mathematical modelling of real-life problems usually results in fractional integro differential equations and various other problems involving special functions of

mathematical physics as well as their extensions and generalizations in one or more variables [4, 21, 22, 23, 24, 33, 45].

3 Preliminaries

In this section, we recall some important notation, definitions, and primary results together with references.

Definition 1. [45] The Riemann-Liouville fractional integral of order $\nu > 0$ of a function f is defined as

$$\begin{aligned} J^{\nu}h(t) &= \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}h(s)ds, \qquad t>0, \quad \nu \in \mathbb{R}^+, \\ J^{0}h(t) &= h(t), \end{aligned}$$

where $\mathbb{R}^+$ is the set of positive real numbers.

Definition 2. [35] The Riemann-Liouville derivative of order ν with the lower limit zero for a function $h : [0, \infty) \longrightarrow \mathbb{R}$ can be written as

$$D^{\nu}h(t) = \frac{1}{\Gamma(1-\nu)}\frac{d}{dt}\int_0^t \frac{h(s)}{(t-s)^{\nu}}ds, \ t>0, \ 0<\nu<1. \tag{1}$$

Definition 3. [33] The Caputo derivative of order ν for a function $h : [0, 1) \longrightarrow \mathbb{R}$ can be written as

$$D^{\nu}h(x) = \frac{1}{\Gamma(1-\nu)}\int_0^x \frac{h'(t)}{(x-t)^{\nu}}dt, \ x>0, \ 0<\nu<1.$$

Definition 4. [33] The fractional derivative of $h(x)$ in the Caputo sense is defined by

$$\begin{aligned} {}^{c}D^{\nu}h(x) &= J^{n-\nu}D^{n}h(x) \\ &= \begin{cases} \frac{1}{\Gamma(n-\nu)}\int_0^x (x-t)^{n-\nu-1}\frac{d^n h(t)}{dt^n}dt, & n-1<\nu<n, \\ \\ \frac{d^n h(x)}{dx^n}, & \nu = n, \end{cases} \end{aligned} \tag{2}$$

Definition 5. [45] The Riemann-Liouville AB-derivative of order $0 < \nu \leq 1$ of a function $u \in C([0,T])$ is normally defined as

$${}^{AB}D^{\nu}_{0^+}u(t) = \frac{\beta(\nu)}{1-\nu}\frac{d}{dt}\Big(\int_0^t u(s)E_{\nu}[-\nu\frac{(t-s)^{\nu}}{1-\nu}]ds\Big). \tag{3}$$

Definition 6. [45] The Caputo AB-derivative of order $0 < \nu \leq 1$ of a function $u \in C([0,T])$ is normally defined as

$${}^{ABC}D^{\nu}_{0^+}u(t) = \frac{\beta(\nu)}{1-\nu}\int_0^t u'(s)E_{\nu}[-\nu\frac{(t-s)^{\nu}}{1-\nu}]ds. \tag{4}$$

Definition 7. [4] The Hadamard derivative of fractional order $\nu > 0$ for a continuous function $h : [1,\infty) \longrightarrow \mathbb{R}$ is defined as

$$D^{\nu}h(t) = \frac{1}{\Gamma(n-\nu)}\Big(t\frac{d}{dt}\Big)^n\int_1^t \Big(\log\frac{t}{s}\Big)^{n-\nu-1}h(s)\frac{ds}{s},\ n-1<\nu<n. \tag{5}$$

where $n = [\nu]+1$, and $[\nu]$ denotes the integer part of real number ν.

Definition 8. [4] The Hadamard fractional integral of order ν for a continuous function h is defined as

$$I^{\nu}h(t) = \frac{1}{\Gamma(\nu)} \int_1^t \Big(\log \frac{t}{s} \Big)^{\nu-1} h(s) \frac{ds}{s}, \ \nu > 0.$$

Lemma 1. *[2] Let $n-1 < \nu \leq n, n \in \mathbb{N}$ and $u \in C^n([J])$. Then*

$$(I_1^{\nu} D_1^{\nu} u)(t) = u(t) - \sum_{k=0}^{n-1} \frac{u^(k)(1)}{\Gamma(k+1)} (\log t)^k.$$

Lemma 2. *[2] For all $\mu > 0$ and $\alpha > -1$*

$$\frac{1}{\Gamma(\mu)} \int_1^t \Big(\log \frac{t}{s} \Big)^{\mu-1} (\log s)^{\alpha} \frac{ds}{s} = \frac{\Gamma(\alpha+1)}{\Gamma(\mu+\alpha+1)} (\log t)^{\mu+\alpha}.$$

Lemma 3. *[45] (Gronwall-Bellman's Inequality). Let $u(x)$ and $f(x)$ be nonnegative continuous functions defined on $J = [\alpha, \alpha+h]$ and c be a nonnegative constant. If*

$$u(x) \leq c + \int_{\alpha}^{x} f(s)u(s)ds, \ x \in J,$$

then

$$u(x) \leq c \ \exp\Big(\int_{\alpha}^{x} f(s)ds \Big), \ x \in J,$$

Theorem 4. *[33] Suppose $\nu > 0$, $\tilde{a}(t)$ is a nonnegative function locally integrable on J and $\tilde{g}(t)$ is a nonnegative, nondecreasing continuous function defined on*

$\tilde{g}(t) \leq M, t \in J$, *and suppose* $u(t)$ *is nonnegative and locally integrable on* J *with*

$$u(t) \leq \tilde{a}(t) + \tilde{g}(t) \int_0^t (t-s)^{\nu-1} u(s) ds, \ t \in J.$$

Then

$$u(t) \leq \tilde{a}(t) + \int_0^t \Big[\sum_{m=1}^{\infty} \frac{(\tilde{g}(t)\Gamma(\nu))^m}{\Gamma(m\nu)} (t-s)^{m\nu-1} \tilde{a}(s) \Big] ds, \ t \in J.$$

Lemma 5. *[33] (Ascoli-Arzela theorem) Let* $S = \{s(t)\}$ *is a function family of continuous mappings* $s : J \longrightarrow X$. *If* S *is uniformly bounded and equicontinuous, and for any* $t^* \in J$, *the set* $\{s(t^*)\}$ *is relatively compact, then there exists a uniformly convergent function sequence* $\{s_n(t)\}(n = 1, 2, ..., t \in J)$ *in* S.

Theorem 6. *[45] (Banach's fixed point theorem) Let* (X, d) *be a nonempty complete metric space with* $T : X \longrightarrow X$ *is a contraction mapping. Then map* T *has a fixed point* $x^* \in X$*such that* $Tx^* = x^*$.

Theorem 7. *[33] (Krasnoselskii's fixed point theorem) Let* M *be a closed convex and nonempty subset of a Banach space* X. *Let* A, B *be two operators such that:*

1. $Ax + By \in M$ *whenever* $x, y \in M$.

2. A *is compact and continuous.*

3. B *is a contraction mapping.*

Then there exists $z \in M$ *such that* $z = Az + Bz$.

Theorem 8. *[45] (Krasnoselskii-Schaefer fixed point theorem). Let* S *be nonempty, closed, bounded and convex subset of a real Banach space* X *and Let* T_1 *and* T_2 *be*

operators on S satisfying the following conditions

1. T_1 is contraction on S.

2. T_2 is completely continuous on S.

Then either

(I) There exists a $x \in S$ such that $T_1x + T_2x = x$, or

(II) The set $\epsilon = \{u \in X : \lambda T_1(\frac{u}{\lambda}) + \lambda T_2(u)\}$ is unbounded for $\lambda \in (0, 1)$.

Theorem 9. *[33, 45] (Schaefer's fixed point theorem) Let $T : X \longrightarrow X$ be a completely continuous operator. If the set $S = \{x \in X : x = \lambda Tx\}$ for some $\lambda \in (0, 1)$ is bounded, then, T has fixed points.*

Theorem 10. *[45] (Schauder's fixed point theorem) Let X be a Banach space and $B \subset X$ be a convex, closed and bounded set. If $\Omega : B \longrightarrow B$ is a continuous operator such that $\Omega B \subset X$, ΩB is relatively compact, then Ω has at least one fixed point in B.*

4 Problem Statement

The recent studies on fractional differential equations indicate that a variety of interesting and important results concerning existence and uniqueness of solutions, stability properties of solutions, analytic and numerical methods of solutions of these equations have been obtained and the serge of investigating more and more results is underway. The tools of fractional calculus have played a significant role in improving the modeling techniques for several real world problems. One of the characteristics of operators of fractional-order is their nonlocal nature accounting for the hereditary properties of many phenomena and processes involved.

The objective of this work is to investigate and develop boundary and initial value problems involving Caputo, Riemann-Liouville, Hadamard, Atangana-Baleanu, Hilfer fractional derivatives and integrals. We establish some new existence, uniqueness, stability and controllability results for the given problems by means of classical fixed point theorems and Banach contraction principle. All chapters concludes with some illustrative examples.

5 Methodology to be Adopted

My area of research is theoretical in nature. Mostly, I will be referring the text books and research papers of various researchers who have developed this area of research. I would also like to consult with my thesis supervisor time to time to discuss the doubts. I would also like to participate in conferences, seminars,

workshops in my area of research, which will help me to strengthen my knowledge in my field of research.

6 Importance of Study

The area of my research is highly applicable in several real-world phenomena of different fields of applied sciences and engineerings like statistical physics, polymer physics, optics, medicine, economics, chemical engineering, signal processing, electrochemistry, electromagnetic,visco-elasticity, porous media, rheology, proteins, astrophysics, chaotic dynamics, and acoustic control, etc. One of the important class of fractional differential and integral equations is known as fractional integro-differential equations (FIDEs). Numerous real-world problems can be modeled by FIDEs in the sciences and engineering disciplines. Here, it is remarkable that integro-differential equations are increasingly used to model a radiation energy intensity along the ray in the emitting, absorbing, and scattering medium. There have been traces of very important applications of the said equations in fluid mechanics and plasma physics.

This proposal will present a broad view of the subject and relates major results to theoretical sides for the mentioned problems.

7 Objectives of Research

- Study the existence and uniqueness of the solutions of nonlinear fractional integro-differential equations.
- Study the stability and data dependence of the solutions of nonlinear fractional integro-differential equations.
- Study the exact controllability and continuous dependence of different types of nonlinear fractional integro-differential equations.
- Study the mild solutions for nonlinear fractional integro-differential equations.
- Develope the generalized techniques to analyze fractional integro-differential equation with different conditions.

8 Proposed Work Plan

Following is the Year-wise plan of work and target to be achieved during my research period:

Year 1: Course work: I would like to complete the course work which is declared by SRTMU Nanded. Then after, I would like to read the following text books to develop necessary background for my research:

(i) Miller, K. and Ross, B. (1993), An Introduction to the Fractional Calculus and Differential Equations, John Wiley, New York.

(ii) Lakshmikantham, V. and Rao, M. (1995), Theory of Integro-Differential Equations, Gordon & Breach, London.

(iii) Zhou, Y. (2014), Basic Theory of Fractional Differential Equations, Singapore: World Scientific.

(iv) Samko, Stefan G. (1987), Fractional Integrals and Derivatives, Theory and Applications, Minsk; Nauka I Tekhnika.

Year 2: I would like to read some related works.

(i) I will read some related text books.

(ii) I will start reading some research papers of various researchers in the area of my research topic and then decide the problem to solve for Ph.D.

(iii) I'm going to concentrate on my research problem.

Year 3: In the domain of my study topic, I plan to read and write some research articles.

Year 4: I will write and submit my thesis.

9 Thesis Layout

- **Chapter 1:** Introduction and Overview.
  1.1- Introduction.
  1.2- Review of Literature and Development in the Subject.
  1.3- General Notations.
- **Chapter 2:** Nonlinear Riemann-Liouville Fractional Integro-Differential Equations.
- **Chapter 3:** Nonlinear Hilfer Fractional Integro-Differential Equations.
- **Chapter 4:** Nonlinear Caputo Fractional Integro-Differential Equations.
- **Chapter 5:** Nonlinear Caputo-Atangana-Baleanu Fractional Integro-Differential Equations.
- **Chapter 6:** Nonlinear Hadamard Fractional Integro-Differential Equations.
- **Chapter 7:** Conclusions and Future Work.

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